Ballpark estimates

How to impress your date and amaze your friends with off-the-cuff answers to questions of magnitude

by David Halliday

SOME PROBLEMS OF PHYSICS involve calculations of the highest possible precision. Many problems, however, call for only an approximate answer. Physicists pride themselves on being able to solve such "order-of-magnitude problems" quickly by breaking them down into their components and making appropriate common-sense estimates.

Here's a typical problem:

> On average, how many atoms of rubber are worn from an automobile tire every time the wheel goes around?

Problems of this kind are often called "Fermi problems" after the great physicist Enrico Fermi, who was a great practitioner of the craft of proposing them and solving them quickly and cleverly.

No doubt you have a few questions.

Yes, I do. Does this problem have any practical significance?

Probably not. Although the problem is an interesting link between the worlds of the very small (the atom) and the very large (the automobile), its real purpose is to help you understand how to make estimates.

But there are no numbers. How can we even start?

We have to estimate the starting numbers—the radius of a tire, the amount of wear ...

But that's just guessing! How can we possibly arrive at an accurate answer?

If by "accurate" you mean an answer good to three significant figures, you're right. But in a problem of this kind, "accurate" means "within a factor of ten either way"—that is, over or under. Actually, it's hard to be that far wrong in estimating the input data.

I get it. Where do we start?

We start with a plan. We'll estimate the volume of rubber worn from the tire and then divide by the volume of an atom. That will give us our answer. Let's deal with the tire first.

Okay. But I don't see any way to guess what volume of rubber is worn from the tire every time the wheel goes around.

We can get an estimate by guessing the volume of rubber worn during the life of the tire and then figuring out how many revolutions the wheel makes during that time. Dividing will give the volume of rubber lost per turn.

Let R be the outer radius of the tire, W the width of the tread, h the depth of wear, and L the distance traveled during the life of the tire. The number of turns N is the total distance traveled divided by the length of the tire's circumference:

$$N = \frac{L}{2\pi R},$$

in which $2\pi R$ is the circumference of the tire. The volume of worn rubber V is the volume of a cylinder of thickness h:

$$V = (2\pi R)Wh.$$

The volume worn per turn is then

$$V_t = \frac{V}{N} = \frac{(2\pi R)Wh}{L/2\pi R} = \frac{(2\pi R)^2 Wh}{L} = \frac{40R^2Wh}{L}.$$

Notice that we've replaced π^2 by 10, which is certainly close enough for our purposes.

But there's no need to replace π^2 by 10. My calculator shows 9.87.

You might feel that you're improving the precision of our answer by doing that, but you're not. Our other estimates will be so approximate that such precision is misplaced. Not only that, 10 is a much simpler number to deal with.

I accept that. What next?

We've already made great progress. We've reduced part of the problem to quantities we can estimate. We'll do that soon. Meanwhile, let's think about atoms.

I've been wondering about that. What is a "rubber atom," anyway? I'm sure you won't find it in the periodic table!

You're right, of course. Rubber is made up of long chain molecules formed from carbon, hydrogen, and oxygen atoms. We're interested here only in a sort of generic atom, whose radius we label r.

I see. Then the volume V_a of the generic atom would be the volume of a sphere of radius r, or $(4\pi/3)r^3$. Right?

You could say that. It's a little better (and simpler) to put the volume at $(2r)^3$—that is, the cube of the diameter. That treats the atoms as little cubes and makes some allowance for the empty space between them.

Now we divide to find our answer. Right?

Right. The number of atoms worn away per turn is

$$n = \frac{V_t}{V_a} = \frac{40R^2Wh}{L(2r)^3} = \frac{5R^2Wh}{Lr^3}.$$

Now we're ready for our estimates. Let's take them one at a time:

R (tire radius) = about 1 ft or 30 cm or 3/10 m,

W (tread width) = about 4 in or 10 cm or 1/10 m,

h (depth of tread wear) = about 1/6 in or 4 mm or 4/1000 m,[1]

L (tire life) = about 50,000 mi or $8 \cdot 10^7$ m,

r (radius of an atom) = about 10^{-10} m.[2]

In putting these numbers into the above expression for n, we must be careful to choose units consistently. Using meters, we find

$$n = \frac{5 \cdot 3 \cdot 3 \cdot 4}{10 \cdot 10 \cdot 10 \cdot 1000 \cdot 8 \cdot 10^7 \cdot 10^{-30}}.$$

Shall I work this out on my calculator for you?

No! It's a point of honor not to use a calculator when solving Fermi problems. Let's rewrite this equation by collecting the integers and the powers of ten:

$$n = \left(\frac{5 \cdot 3 \cdot 3 \cdot 4}{8}\right) \cdot 10^{17}.$$

You can easily see that the number in the parentheses is about 20, so that $n = 2 \cdot 10^{18}$ atoms per turn.

Shouldn't we round that off to 10^{18} atoms per turn?

Yes, indeed. The "2" isn't justified by the precision of our estimates.

So—

When someone asks the "tire question" at a party (and it never fails to come up, believe me!), you can now gaze at the ceiling for a few minutes and say: "About ... 10^{18} atoms per turn, more or less." That's how quickly Fermi himself solved problems like this one!

Try your hand at finding ballpark estimates for these Fermi problems.

1. The population of Boston in 1980 was about 560,000. How many high school teachers were there in that city in that year?

2. How many gallons of gasoline are consumed each year in the United States by private automobiles? ◘

Adapted from the forthcoming book Essentials of Physics *by David Halliday, Robert Resnick, and John Merrill with permission of the publisher, John Wiley & Sons, Inc.* David Halliday *is professor emeritus of physics at the University of Pittsburgh.*

[1] *You* might estimate the depth of tread wear to be 1/2 in (12 mm). If so, your calculations will be slightly different. That's okay—these are estimates.—*Ed.*

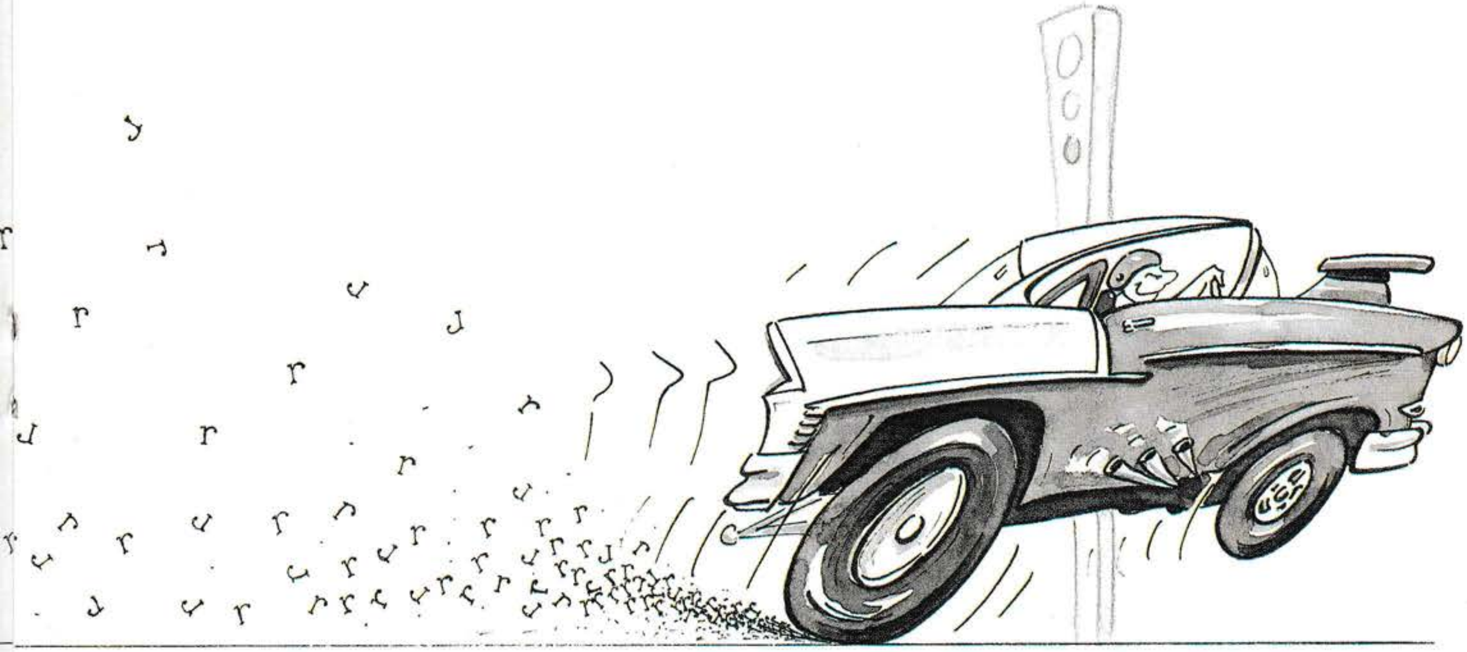
[2] Physicists always use this as an estimate of the radius of an atom. It's a good number to know. (The radius of a nucleus, by the way, is estimated to be 10^{-15} m.)—Ed.

Large or small?

Do you consider the answer to the tire problem (10^{18} atoms/turn) large or small? No answer is possible until you've answered the necessary auxiliary question: Large or small relative to what? As a pure number, 10^{18} seems large. It's 10,000,000 times greater than the number of stars in the Milky Way galaxy, for example.

But the problem deals with 10^{18} *atoms*, not 10^{18} as a pure number. This number of atoms is about 10,000,000 times *greater* than the number of atoms in a typical small bacterium but about 10,000,000 times *smaller* than the number of atoms in a glass of water.

Our conclusion: You can only compare physical quantities of the same kind. There are no absolute standards of "large" or "small."



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